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Discrete Symmetry in Lorentzian Spaces

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Contents

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In this essay, we construct a mathematical framework to generalise the use of reflection groups in classifying discrete symmetries of Lorentzian spaces, inspired by both the immense mathematical applicability of familiar Coxeter theory and potential applications to discrete models of spacetimes. With this goal, we present a generalisation of the notion of crystallographic symmetry, and argue its necessity. Utilising this generalisation, we show how properties of reflection groups and mirror hyperplanes in Euclidean spaces turn out to be vastly different in Lorentzian spaces.

Statement of original research

Chapter 2 is a review of various results in the study of Euclidean Coxeter groups [1]. Chapter 3 clarifies notions about crystallographic symmetry and proposes a definition most suited for a study of reflection groups in general spaces. Non-Euclidean spaces are investigated in Chapter 4, which also includes original results about properties of reflections in indefinite spaces. Chapter 5 further proposes results about reflection groups in indefinite spaces and begins arguments about how such groups divide the space. All figures are original unless stated.

1 Introduction

We begin this introduction quoting the introduction of Donald Coxeter's PhD thesis -

> *Although it is unnecessary, from a practical point of view, the human weakness of a mathematician compels him to examine the general case, although extraordinarily complicated. ... The only excuse for this part of the work must be its intrinsic beauty.* [2]

In a similar vein, this essay is motivated both by practical pursuits, introduced first, and by the curiosity to examine and systematically study what lies beyond the known. To begin with the more practical, lattices and regular structures play a foundational role in a wide range of problems arising in pure mathematics [3, 4], physics, and chemistry. For them, the mathematical formalism of lattices and their discrete symmetry groups appears as the fundamental connecting piece, providing us with some of the most exciting connections between seemingly unrelated disciplines. Aside from the many successful applications of lattices as modelling tools in the field of discretised numerical simulations in physics, the study of lattices and their symmetries is of integral importance for classifying crystalline (and quasi-crystalline) structures that can occur within materials [5–7]. As an example, the lattice structure describing the arrangements of atoms inside materials and crystals gives rise to the distinctive properties of the material [8]. By studying such lattices and their associated symmetry groups, physicists can gain insights into the fundamental properties of matter and develop a deeper understanding of the world around us.

We begin this essay by considering lattice symmetries arising in Euclidean space (Chapter 2). Motivated by the need to describe both point symmetries (symmetries having fixed points) and translation symmetries (having no fixed points), we'll describe lattice symmetries through the lens of reflection groups, and more generally, Coxeter groups [1, 9]. This provides a comprehensive summary of possible discrete symmetries that can arise. By exploring such groups, we can not only classify the allowed three-dimensional crystal-like symmetry [10] but also lay out a formalism for describing regular and semi-regular discretisations in Euclidean spaces of arbitrary dimension. Remarkably, this mathematical formalism for crystalline discrete symmetry also makes an appearance in the classification of finite semi-simple Lie Algebras [11].

This foundation serves as a first step for our subsequent investigation into the corresponding theory of discrete groups in non-Euclidean spaces (Chapter 4), such as Hyperbolic space, which can be seen as an embedded surface in Minkowski space. As we will see, this space is much less restrictive, allowing for more regular and quasiregular discrete symmetries [12, 13]. Groups describing hyperbolic reflection symmetry crop up in string theory and pure mathematics more generally through Kac-Moody algebras [14, 15] (which are seen as generalisations of the finite Lie theory) and vertex operator algebras. Further, there exist tensor network models of AdS space that can accurately recover properties expected from holography when such a discretisation of AdS space is sufficiently symmetric [16, 17]. Such relevance of hyperbolic tilings in both mathematics and physics provides us with a strong motivation for exploring the effect discrete reflection groups may have, not just in a hyperbolic subspace of Minkowski space, but rather in a general Lorentzian space. Indeed, considering discrete subgroups of the full set of possible Lorentzian isometries could allow for both previously unseen mathematical beauty and practical discrete models of spacetime. Such a study is the main goal of this line of research, with this report being the first step.

While seeing various examples of reflection groups in the aforementioned spaces, we begin noticing a fundamental property that some of these groups have while others don't: does the group (potentially an infinite group) divide the space that it's acting on into region regions of finite and non-zero volume? We denote such a group as "kaleidoscopic" (Chapter 3). Note that this definition depends on the space one is considering; the same groups can be kaleidoscopic in one space but not another, as we shall see. In surrounding literature, the term "crystallographic" refers to the symmetry groups (and subgroups) of lattices, such as the groups describing symmetries of crystals and materials. We'll motivate our definition of kaleidoscopic, in terms of the volume of a group acting on a space, both by geometric and algebraic arguments, and show kaleidoscopic symmetry is equivalent to the more standard crystallographic symmetry *in Euclidean spaces*. Further, kaleidoscopic symmetry is more applicable to curved spaces, where the notion of lattices is no longer sensible, but the notion of reflection groups still is.

Finally, our definition of kaleidoscopic symmetry will be integral in our goal of understanding reflection groups in Lorentzian spaces (Chapter 5). We will show that such groups are often infinite unless significant constraints are imposed. We'll also discuss preliminary arguments about allowed kaleidoscopic Lorentzian groups, stemming from non-intuitive properties of reflections that do not generalise from the Euclidean case. In future work, we hope to either definitively prove stronger restrictions on Lorentzian kaleidoscopic symmetry or construct previously unclassified reflection groups and discretisations; both outcomes we regard as incredibly insightful in better shaping both a pure and practical understanding of indefinite spaces and their symmetries.

2 Euclidean Discrete Symmetry

The goal of this chapter is to motivate the study of discrete symmetry, which we will do initially by considering symmetries of lattices. We motivate why the best formalism for studying these symmetry groups is with reflection groups. We then describe further mathematical and geometric constructions, such as Coxeter groups, Kaleidoscope groups, and fundamental domains, that arise from reflection groups.

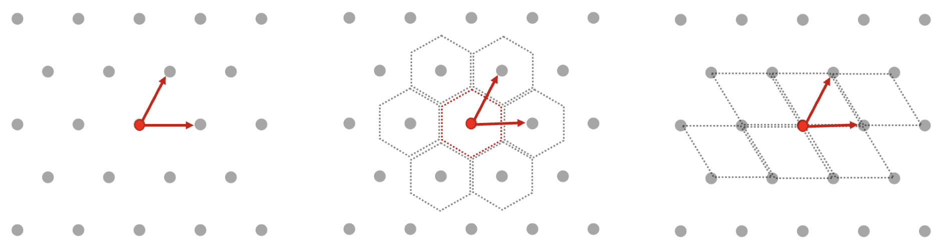
Remark. We take V to be a vector space with an inner product, written $\langle \cdot, \cdot \rangle$. In this chapter, V is a Euclidean vector space and so the inner product is the usual Euclidean inner product.

2.1 Lattices

Lattices play an incredibly important role in many facets of mathematics and physics, including but not limited to number theory [18], algebra and topology [19], probability [20], invariant theory [21], cryptography [22, 23], modeling dynamics [24–26], numerical methods, Fourier analysis [27], solid state physics, and chemistry. Furthermore, understanding the geometries of lattices can be useful for various tasks, like determining physical properties (e.g. energy bands) of materials to providing mathematical classifications (e.g. semi-simple Lie theory). With this goal, we introduce the formal language used to describe lattices and their symmetries.

There can be differing definitions in the literature depending on one's field[1], but in this essay, a lattice is always a **Bravais lattice**, which is a

[1]In the study of crystals and materials - crystallography - a "lattice" does not have to be Bravais; it can be a larger structure of multiple points called a "basis" which is then translated across the space by a Bravais lattice to form a crystal. The honeycomb "lattice" and Kagome "lattice" are examples of this; neither are Bravais.



(a) Hexagonal Lattice (b) Wigner-Seitz unit cell (c) A less symmetric unit cell

Figure 1: The Hexagonal Lattice with translation vectors and two unit cells

discrete subset of a real n-dimensional vector space V, defined by a set of n basis vectors $\vec{a_i} \in V$ by

$$\Lambda = \{\vec{x} \in V \mid \vec{x} = n_i \vec{a_i} \text{ for } n_i \in \mathbb{Z}\}. \tag{1}$$

This essay is concerned with discrete symmetry, so we can ask what group of linear transformations of V also preserve Λ. As Λ is a discrete subset of V, the group of transformations preserving Λ is a discrete subgroup of all automorphisms of V. The full symmetry group of the lattice can be further analysed by considering what kind of regions of the space, when translated by the vectors $\vec{a_i}$, tessellate the space. Such a region is called a unit cell. Two examples of different unit cells for the Hexagonal Lattice are provided in Figure 1, with translation vectors shown in red.

Clearly, the choice of a unit cell is not unique, and thus different choices can be more or less symmetric. Certainly, a cell cannot have *more* symmetries than the lattice, as it tiles the lattice, so any symmetry of the cell has to be a symmetry of the lattice. However, the converse does not hold: the unit cell does not need to have all non-translational symmetries of Λ. In fact, there is a choice of unit cell with the same non-translational symmetries as Λ, called the **Wigner-Szeitz cell** (or **Voronoi cell**), given by

$$C(\Lambda) = \{\vec{x} \in V \mid \langle \vec{x}, \vec{x} \rangle < \langle \vec{x}, \vec{\ell} \rangle \text{ for all } , \vec{0} \neq \vec{\ell} \in \Lambda\}. \tag{2}$$

Lemma 1. *The unit cell defined in Equation* (2) *admits the same non-translational symmetries as* Λ.

Proof. For some non-translational symmetry operation R of the lattice (meaning $R\Lambda = \Lambda$ and R preserves norms) and $\vec{x} \in C(\Lambda)$, we have

$$\langle R\vec{x}, R\vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle < \langle \vec{x}, \vec{\ell} \rangle = \langle \vec{x}, \text{adj}(R)\vec{k} \rangle = \langle R\vec{x}, \vec{k} \rangle, \tag{3}$$

where $\vec{\ell}, \vec{k} \in \Lambda$ as $\mathrm{adj}(R)\vec{k} = R^\top \vec{k} = R^{-1}\vec{k} = \vec{\ell} \in \Lambda$ since R is a non-translational symmetry of Λ and thus orthogonal. Thus $R\vec{x} \in C$. ■

Note that $C(\Lambda)$ contains exactly one lattice point, namely $\vec{0}$. The group generated by all non-translational R, as above, forms a symmetry group of $C(\Lambda)$, which we call a **point symmetry group**. We can see in Figure 1b that the Wigner-Seitz unit cell for the hexagonal lattice is, unsurprisingly, a hexagon, with 6-fold symmetry for its point group (dihedral group D_6); indeed the entire hexagonal lattice has this point symmetry as well. However Figure 1c shows a choice of unit cell with only order-2 rotational symmetry, a strict subgroup of the full group of point symmetries of the lattice.

These notions are precisely related to **point symmetries**, which are the symmetry transformations that leave at least one point fixed, such as inversion (leaves the center point fixed), rotation about an axis (leaves the axis fixed) and mirror plane reflection (leaves the plane fixed). Translations are not point symmetries as they have no fixed points. Thus all the symmetries of the lattice are given by compositions of translations by lattice vectors and elements of the point symmetry group.

We can now ask the question - for some given point symmetry group, does there have to exist a lattice that has a unit cell with this symmetry group; or equivalently, can our given point group be a subgroup of the full lattice symmetry group? For example, as will be shown below, no lattice can have pentagonal symmetry (D_5) as a subgroup, while we have already seen a lattice with hexagonal symmetry (D_6). The class of groups for which at least one such lattice exists we shall call **crystallographic**[2]. In solid state and condensed matter physics, materials are modeled as atoms or electrons on lattice structures, meaning the point symmetry groups have to be crystallographic. The properties of the materials, such as energy band structure, ground and excited states, magnetisation, etc are all dependent on the W-S unit cell, and therefore on the point symmetries of the lattice. Beyond this, non-crystallographic symmetry groups appear in describing certain special classes of materials called quasi-crystals [28–31], again displaying remarkable properties that depend on the underlying discrete symmetry. Thus, the question of classifying crystallographic groups, i.e. groups that can be decomposed into a lattice translation group and a point symmetry group, is an integral question in mathematics, geometry, and physics. The answer to

[2] There is a possible point of confusion: crystallographic here can denote both a property of the point symmetry group or the full symmetry group of the lattice.

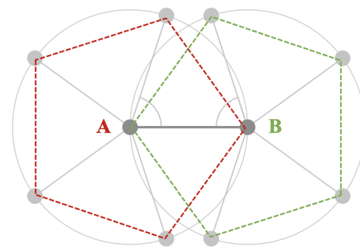


Figure 2: Incompatibility of a lattice with 5-point Symmetry. Modified from Baake Grimm *Aperiodic Order*

this question shall lead us to rich and exciting mathematics, as well as a better understanding of the geometry and the kinds of discrete structures and symmetry groups a given space can admit.

First, let us see a brief example illustrating why classifying crystallographic groups is not trivial. Figure 1b illustrates a 6-fold point symmetry group (the symmetries of the hexagon) combined with a translation group (translating by the two lattice vectors shown in red) in two dimensions. This is equivalent to saying that regular hexagons can tile the plane, or that some lattice has a hexagonal unit cell. However, this *cannot* happen in a Euclidean space for a 5-fold symmetry group: regular pentagons cannot tile the plane; there are no two-dimensional lattices with pentagonal W-S unit cell.

Claim. *A regular tiling of the plane cannot have five-fold symmetry.*

Proof. We can show this by contradiction. Assume we can and consider a 5-fold point symmetry translated to each point of some planar Bravais lattice, which we presume to exist, as shown in Figure 2. Take two nearest-neighbor points A and B; wlog, take the distance between them to be unit (this only corresponds to a rescaling of the lattice). Because of the 5-fold symmetry at A, there must also be four other points in the lattice, which together with B, form a regular pentagon around A. These 4 points are $B \equiv (\cos\frac{0\pi}{5}, \sin\frac{0\pi}{5})$, $(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5})$, $(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5})$, $(\cos\frac{6\pi}{5}, \sin\frac{6\pi}{5})$, and $(\cos\frac{8\pi}{5}, \sin\frac{8\pi}{5})$. Conversely, because of the 5-fold symmetry around B, the points $A \equiv (1+\cos\frac{5\pi}{5}, \sin\frac{5\pi}{5})$, $(1+\cos\frac{3\pi}{5}, \sin\frac{3\pi}{5})$, $(1+\cos\frac{\pi}{5}, \sin\frac{\pi}{5})$, $(1+\cos\frac{9\pi}{5}, \sin\frac{9\pi}{5})$, and $(1+\cos\frac{7\pi}{5}, \sin\frac{7\pi}{5})$ must also be lattice points. However, the distance between $(1+\cos 3\pi/5, \sin 3\pi/5)$ and $(\cos 2\pi/5, \sin 2\pi/5)$ is less than 1, and so we've found two lattice points closer together than A and B. This process could be repeated indefinitely, leading to a contradiction. ■

Indeed, as we shall see later, the lack of 5-fold symmetry in lattices generalises to any dimensions. Specifically, there are no crystals (Bravais lattices of atoms in three dimensions) that have a W-S unit cell with five-fold symmetry (e.g. regular dodecahedron, pentagonal prism, etc).

To further progress in formalising crystallographic symmetry in Euclidean space and beyond, we will begin discussing *reflection groups*. The reason for this is twofold: Firstly, the elements of the point groups must be composed of non-translational isometries of the Euclidean vector space. All of these transformations can be written as compositions of appropriate (non-parallel) reflections; Secondly, the notions of point groups and translation groups seem to be somewhat at odds here, as the group operation for point symmetries is composition, while for translation it is addition. This can be rectified by viewing translations as a composition of affine reflections about two parallel planes[3].

2.2 Reflections and Coxeter Groups

Figure 3: *Contrast (Order and Chaos)* M.C. Escher, 1950

Polygons and polyhedra have long fascinated humanity throughout time, both by appealing to our penchant for artistic fascination as well as providing a basis to describe structure and order that occurs in nature, as we shall see when further discussing lattices and their unit cells. We will never be

[3] This is also precisely the motivation that leads one to the exciting topic of geometric algebra! The interested reader can find an alternate derivation of allowed crystal symmetries from this formalism in [32].

able to directly observe what analogs (called **polytopes**) exist in higher dimensions. However, that could not stop Coxeter, and many mathematicians before and after him, from attempting to glimpse at what wonderous regular discrete structures may exist in higher dimensions and what mathematics could describe them.

A powerful formalism, at the heart of this essay, for answering this question is to describe such regions in space as being bounded by mirror hyperplanes. Specifically, for $d \in \mathbb{R}$ and a vector $\vec{r}$, one defines a hyperplane as the codimension-1 surface given by $\langle \vec{x}, \vec{r} \rangle = d$. A region in an n dimensional space described by a set of mirrors would then have vertices given by the intersection of n mirrors, edges by the intersection of $n-1$ mirrors, faces by the intersection of $n-2$ mirrors, and so on, assuming the normals of the hyperplanes are linearly independent. Thus such hyperplanes give a natural geometric description of bounded regions in space.

Further, this description admits an equivalent algebraic formulation that allows us to connect the study of discrete regions in space with the study of discrete groups. A given hyperplane $\langle \vec{x}, \vec{r} \rangle = d$ also naturally defines a **reflection** operation about that plane given by

$$\vec{x} \mapsto R(\vec{r})\vec{x} + \frac{2d}{\langle \vec{r}, \vec{r} \rangle}\vec{r} \equiv \vec{x} - 2\frac{\langle \vec{x}, \vec{r} \rangle}{\langle \vec{r}, \vec{r} \rangle}\vec{r} + \frac{2d}{\langle \vec{r}, \vec{r} \rangle}\vec{r}. \tag{4}$$

The normal vector $\vec{r}$ which defines the mirror is called a **root**. Here $R(\vec{r})$ is a reflection about the plane $\langle \vec{x}, \vec{r} \rangle = 0$ passing through the origin. Given a basis for the vector space, $R(\vec{r})$ can also be represented as the matrix

$$R(\vec{r})^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - 2\frac{r_{\beta} r^{\alpha}}{\langle \vec{r}, \vec{r} \rangle}, \tag{5}$$

where such a matrix is commonly called a **Householder matrix**. For $d \neq 0$, the reflection (4) is about a plane not passing through the origin (instead it passes through the point $d\vec{r}/\langle \vec{r}, \vec{r} \rangle$) and is also called an **affine reflection**.

We can thus consider the properties of groups generated by such reflections in mirror planes. For two reflections about mirror planes through the origin, $R_1 = R(\vec{r_1}) \neq R(\vec{r_2}) = R_2$, consider $R_1 R_2$. This will fix the orthogonal complement of the space spanned by $\vec{r_1}, \vec{r_2}$, so it suffices to consider the action of $R_1 R_2$ on this space. This will be a rotation of degree 2θ in the space spanned by $\vec{r_1}, \vec{r_2}$, where θ is the angle between the two roots. If θ is a rational division of the unit circle, i.e. $\theta = 2\pi p/q$ for $\gcd(p, q) = 1$, then

$(R_1R_2)^q = 1$ and the group generated by R_1 and R_2 has order q. Else, this group has infinite order and we could write $(R_1R_2)^\infty = 1$.

Further, consider now affine reflective transformations. Let $g_1 = R(\vec{r})$ be a reflection through about $\langle \vec{x}, \vec{r} \rangle = 0$ and let g_2 be a reflection through a parallel plane $\langle \vec{x}, \vec{r} \rangle = d$. Then g_1g_2 gives a translation, $(g_1g_2)^\infty = 1$, and group generated by g_1, g_2 contains a translation group describing translations by the vector $d\vec{r}/\langle \vec{r}, \vec{r} \rangle$.

We see that this description of discrete symmetry by reflections about mirror planes is incredibly well-suited to our question of symmetries of lattices and crystallographic symmetry, since it describes both point symmetry (mirrors through the origin) and translation symmetry. Indeed what we have discussed have been interpretations of a general **Coxeter group**, which is a group generated by elements g_i such that

$$\begin{aligned} (g_i)^2 &= 1, \\ (g_ig_j)^{m_{ij}} &= 1 \text{ for } i \neq j, \text{ where } m_{ij} = m_{ji} \in \mathbb{N}. \end{aligned} \tag{6}$$

As we have seen above, m_{ij} are allowed to be ∞.

2.3 Kaleidoscopes

Let us now proceed to investigate the symmetries of a lattice purely through affine Coxeter groups, i.e. generated by reflections about planes both through and displaced from the origin. Viewing this not as a group of transformations, but rather as an (infinite) set of mirrors gives us a **kaleidoscope**. We can then formulate questions about what region of space one "sees" when looking into the kaleidoscope, and how to recover the original lattice from this "fundamental region." Following this examination, we can investigate the relationship between such fundamental regions, point groups, and kaleidoscopes to formulate key results about crystallographic groups, i.e. point groups that can be combined with a lattice.

Remark. A kaleidoscope will denote a collection of mirror hyperplanes in a space. We will also refer to "kaleidoscope groups" to mean the groups of reflections about the mirror hyperplanes in a kaleidoscope.

For a lattice Λ, we must first find its corresponding root vectors, as these will determine its mirrors of symmetry and ultimately the kaleidoscope. A root of the lattice will be a vector $\vec{r}$ such that $R(\vec{r})\Lambda = \Lambda$. If $\vec{r} \notin \Lambda$ then no rescaling of $\vec{r}$ is in Λ, and so for $\vec{\ell} \in \Lambda$, $R(\vec{r})\vec{\ell} \notin \Lambda$ (unless $\vec{r}$ is perpendicular to the entire lattice, meaning that $R(\vec{r})$ acts on Λ trivially). Thus all roots of the lattice will be lattice vectors themselves. Further, a **primitive vector** $\vec{v} \in \Lambda$ has no $n > 1$ s.t. $\vec{v}/n$ is also a vector in the lattice. Mirror planes do not depend on the length of the root, and thus primitive vectors of the lattice give us an intuitive ansatz for the roots of the lattice.

The **Gram matrix** of a lattice is given by $G(\Lambda)_{ij} = \langle \vec{a_i}, \vec{a_j} \rangle$, where the vectors $\vec{a_i}$ define the lattice. Note that if all the entries of $G(\Lambda)$ are rational, then there exists some smallest $\mu \in \mathbb{R}$ such that the lattice $\{\mu a_i\}$ has integer Gram matrix. We call a lattice of this type an **integral lattice**, and from now on will take such lattices to have $\langle \vec{v}, \vec{w} \rangle \in \mathbb{Z}$ for all $\vec{v}, \vec{w} \in \Lambda$. Note that this is important when it comes to defining reflections that preserve a lattice, since if $\langle \vec{v}, \vec{v} \rangle \notin \mathbb{Q}$ then $R(\vec{v})\vec{\ell} \notin \Lambda$ for $\vec{\ell} \in \Lambda$.

Finally, we define the **dual** Λ^* of Λ, which can be done in several equivalent ways.

$$\begin{aligned}
\Lambda^* &= \{\alpha : \Lambda \longrightarrow \mathbb{Z}\} \\
&= \{\alpha \in V^* \equiv \{f : V \longrightarrow \mathbb{R}\} \mid \alpha(\vec{x}) \in \mathbb{Z} \text{ for all } \vec{x} \in \Lambda\} \\
&\cong \{\vec{y} \in V \mid \langle \vec{x}, \vec{y} \rangle \in \mathbb{Z} \text{ for all } \vec{x} \in \Lambda\} \\
&\cong \{\vec{y} \in V \mid \vec{y} = m_i \vec{b_i} \text{ for } m_i \in \mathbb{Z} \text{ for } \vec{b_i} \text{ s.t. } \langle \vec{b_i}, \vec{a_j} \rangle = \delta_{ij}\} \qquad (7)
\end{aligned}$$

We first easily see that integral lattices have $\Lambda \subset \Lambda^*$. Given this, and the fact that Λ, Λ^* are both abelian groups, the quotient group Λ^*/Λ is readily well-defined. This group describes equivalence classes of points in Λ^* within the unit cell of Λ that are separated by vectors in Λ, or, "how nonequivalent" Λ^* and Λ are.

Lemma 2. *For an integral lattice* Λ*,* $|\Lambda^*/\Lambda| = |\det G(\Lambda)|$*. This quantity is called the* ***lattice determinant*** *and written* $\det(\Lambda)$*.*

Proof. Any unit cell of Λ has volume given by $|\det([\vec{a_1}, ..., \vec{a_n}])|$, which can be seen by considering the parallelepiped defined by basis vectors $\vec{a_i}$. Thus $|\det G(\Lambda)|$ is equal to this volume squared. Now scale Λ by some $s \in \mathbb{Z}$. The volume of the unit cell will scale by a volume factor of q^n. Further, the density of Λ^* points within the original volume will increase by q^n as $(b_i, a_i) = 1$ for each i. Thus $|\Lambda^*/\Lambda|$ should be proportional to $|\det G(\Lambda)|$ (i.e. volume squared). If the lattice is self dual ($\Lambda = \Lambda^*$) then $|\Lambda^*/\Lambda| = 1$, and so we have exactly $|\Lambda^*/\Lambda| = |\det G(\Lambda)|$. ■

Theorem 1 ([33, 34])**.** *If $\vec{v} \in \Lambda$ is a primitive root vectors of Λ, then $\langle \vec{v}, \vec{v} \rangle$ must divide $2det(\Lambda)$.*

Proof. Take $\vec{w} := 2\vec{v}/\langle \vec{v}, \vec{v} \rangle$. Have $R(\vec{v})\vec{\ell} = \vec{\ell} - \langle \vec{w}, \vec{\ell} \rangle \vec{v} \in \Lambda$ for any $\vec{\ell} \in \Lambda$ as $R(\vec{v})\Lambda = \Lambda$, so $(\vec{w}, \vec{\ell}) \in \mathbb{Z}$ as $\vec{v}$ is primitive. Thus $\vec{w} \in \Lambda^*$. The order of the coset $\vec{w} + \Lambda$ in Λ^*/Λ is the smallest integer m s.t. $m\vec{w} \in \Lambda$, which is either $\langle \vec{v}, \vec{v} \rangle$ or $\langle \vec{v}, \vec{v} \rangle/2$ as $\vec{v}$ is primitive. The order of an element divides the order of the group (for a finite group), and so $\langle \vec{v}, \vec{v} \rangle$ must divide $2\det(\Lambda)$. ■

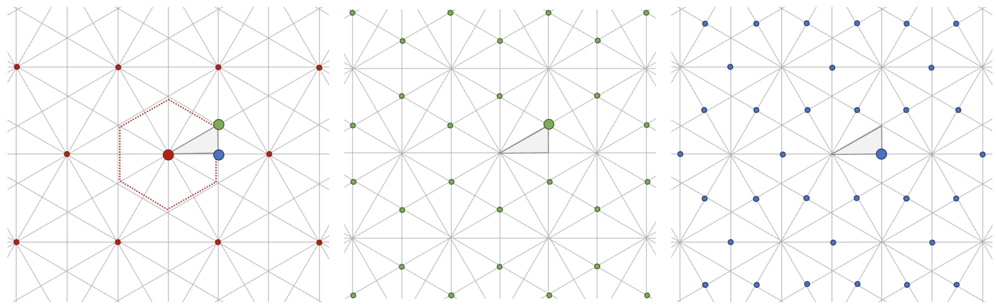
Example. Take the Hexagonal lattice as seen in Figure 1b with basis vectors $\vec{a_1} = (\sqrt{2}, 0)$ and $\vec{a_2} = (\sqrt{2}/2, \sqrt{3}/\sqrt{2})$. This is the smallest scaling of the hexagonal lattice such that the Gram matrix has integer entries. This matrix has determinant 3, and thus to compute the roots of Λ we should consider lattice vectors with norm 1, 2, 3, and 6. There are no root vectors of norm 1 or 3, and so the roots are the vectors

$$(\pm\sqrt{2}, 0) \qquad (\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{\sqrt{2}}) \qquad (0, \pm\sqrt{6})$$

or equivalently, 6 mirror planes passing through the origin, with $\pi/6$ the smallest angle between them.

Once the mirror planes of symmetry (through the origin) for the lattice have been found, "translating this point group" to each lattice point produces a kaleidoscope of infinitely many mirrors in the space. Said more formally, this consists of including affine reflections such that the number of mirrors passing through each lattice point is equal to the number of mirrors in the point group. This is illustrated in Figure 4 for the hexagonal lattice. We see that the mirrors divide the space into infinitely many equivalent triangles, with angles $\pi/6$, $\pi/3$, and $\pi/2$.

For an arbitrary kaleidoscope, let Γ to be the group containing all affine transformations defined by mirrors in a kaleidoscope; i.e. for a mirror hyperplane defined by $\langle \vec{x}, \vec{r} \rangle = d$, the corresponding element $\gamma = \gamma(\vec{r}, d)$ of Γ acts on V by Equation 4. Consider the orbit space $V/\Gamma \equiv \{\gamma V \mid \gamma \in \Gamma\}$, i.e. the set of orbits for the action of Γ on V, not a quotient group. This region $P = V/\Gamma \subset V$ is called the **fundamental region** (or fundamental domain). Equivalently, P is a subset of the full space V where any point in V outside P is equivalent to some point in P by reflections in Γ. We have seen an example of the fundamental region given by the kaleidoscope for the hexagonal lattice in Figure 4, where P is a 30-60-90 triangle.



(a) Fundamental Domain (b) A honeycomb (c) A (different) honeycomb

Figure 4: Kaleidoscope mirrors, fundamental domain, and honeycombs from Hexagonal lattice. The large red, blue, and green points are various vertices of the fundamental domain, and the smaller points are their images under reflection by the kaleidoscope mirrors.

Continuing this example, we note that reflecting the "origin" of the fundamental domain about all the mirrors in the kaleidoscope regenerates the original hexagonal lattice. This can be seen in Figure 4a, where the origin is the red point, corresponding to the mirrors meeting at $\pi/6$. The green and blue points respectively correspond to angles $\pi/3$ and $\pi/2$, and the image of these points under reflections in the kaleidoscope are not lattices (they can't be, because they lack the origin), but rather regular **honeycombs**, i.e. tilings of polygons. Figure 4b is referred to as $\{6\}$ since it is a honeycomb defined by tiling the regular hexagon. Figure 4c is written $\left\{\begin{smallmatrix}3\\6\end{smallmatrix}\right\}$ since it's a quasiregular tiling of a triangle and hexagon[4]. The original lattice in Figure 4a is also the triangular honeycomb $\{3\}$. This method of constructing lattices and honeycombs from a fundamental domain and kaleidoscope group is called the **Wythoff Construction**. I will not go into further detail about these topics, but the interested reader can find more information in Coxeter's *Regular Polytopes* [1] and another of his papers [36].

Note that this method of reconstructing the lattice only works if the fundamental domain has non-zero and finite volume. It is natural now to ask, why did infinitely many mirrors divide the space into a non-zero and finite minimal volume? Can this be expected for any kaleidoscope? As we shall see in the next section and subsequently in Chapter 3, this is precisely

[4]This honeycomb is also called the Kagome Lattice (again, *not* a Bravais lattice). This structure of atomic arrangements describes certain material which have led to incredibly interesting results in condensed matter physics, both theoretical and experimental [35].

related to crystallographic symmetry.

2.4 Crystallographic Restriction

We've seen previously in two dimensions that five-fold symmetry is not crystallographic, meaning no lattice can have a unit cell with pentagonal symmetry. If certain point symmetry groups are not crystallographic, equivalently we expect there to be certain kaleidoscopes that cannot regenerate any lattice or honeycomb. By analysing the allowed root vectors for crystallographic point groups and kaleidoscopes, we can formulate a complete understanding of crystallographic symmetry in Euclidean space. We shall begin with a simple argument based on the properties of reflections discussed in Section 2.2.

Theorem 2. *If a finite Coxeter group is crystallographic, then each m_{ij} can only be $2, 3, 4,$ or 6 for $g_i \neq g_j$ generators of the group.*

Proof. Suppose there was a lattice compatible with two such generating reflections. As we've seen before, the product of these two reflections can be written as the $n \times n$ matrix

$$\begin{bmatrix} \mathrm{Rot}(2\theta)_{2\times 2} & 0 \\ 0 & \mathbb{I}_{n-2} \end{bmatrix}$$

acting on n dimensional Euclidean space, and thus has trace $2\cos(2\theta)+n-2$. However, we have presumed there exists a (Bravais) lattice compatible with such symmetry, and so the above matrix can be written in the basis given by the primitive vectors for the lattice. Because this operation preserves the lattice, all entries of the matrix must now be integers. Since the trace of a matrix is invariant under change of basis transformations, we must have $2\cos(2\theta) + n - 2 \in \mathbb{Z}$, and so $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$. ■

2.5 Fundamental Domains and Coxeter Diagrams

The classification of Coxeter groups and combinatorial arguments about such groups begins with a discussion of the "triangles" in a geometry (currently Euclidean space) that can be suitably defined by a set of mirror planes. Specifically, in an n dimensional space, a "triangle" (more accurately, simplex, as triangles are polygons) can be defined as the intersection of precisely $n+1$

hyperplanes (each having dimension $n-1$) meeting at $n+1$ vertices (where the set of roots for the mirrors have full rank). In two dimensions this is the triangle, in three dimensions the tetrahedron, in four dimensions the pentachoron (also known as the "5-cell" polytope). There is also technically the trivial and degenerate notion of the one-dimensional simplex, which is a line segment with two vertices. Note that this construction generalises the notion of triangularity nicely, as the triangle (2-simplex) has three edges which are lines (1-simplex), the tetrahedron (3-simplex) has four faces which are triangles (2-simplex), and the pentachoron (4 simplex) has five cells which are tetrahedrons (3 simplex).

A set of planes $P_1, \cdots, P_{n+1}$ in an n-dimensional flat space (i.e. $\mathbb{E}^n$ or $\mathbb{R}^{(+p,-q)}$), with each plane given by $\langle \vec{x}, \vec{r_i} \rangle = d_i$ will define a simplex if any choice of n of the $n+1$ root vectors $\vec{r_1}, \cdots \vec{r}_{n+1}$ has full rank n. We can also take $d_i, \cdots, d_n = 0$ so long as $d_{n+1} \neq 0$. These conditions ensure that one of the vertices of the simplex is the origin, and the intersection of any n mirrors is exactly one point defining a unique vertex. Then the simplex S corresponding to such planes in a space V will be given as the intersection of the halfspaces i.e.

$$S = \{\vec{x} \in V \mid \langle \vec{x}, \vec{r_i} \rangle \leq d_i \text{ for all } 1 \leq i \leq n+1\}. \tag{8}$$

Note the choice to define this region using $\langle \vec{x}, \vec{r_i} \rangle \leq d_i$ instead of $\langle \vec{x}, \vec{r_i} \rangle \geq d_i$ is irrelevant as it merely corresponds to replacing each $\vec{r_i}, d_i$ with $-\vec{r_i}, -d_i$ which changes neither the planes nor the inner product $\langle \vec{r_i}, \vec{r_j} \rangle$ between roots. The minimum number of hyperplane mirrors required to define a bounded volume in n dimensions is $n+1$, and thus simplexes are minimal in this regard. Note that this forces the condition that any n of the roots are linearly independent.

In Coxeter theory, recall there is a one-to-one correspondence between mirrors as geometrical objects (hyperplanes) and mirrors as transformations (representing reflections about that mirror). Thus, for a given simplex, we must consider the notion of reflecting a mirror about another mirror, as the corresponding Coxeter group must be closed under such reflections. For the region to be a fundamental domain, the new mirrors generated upon reflection of a plane bounding the region by another plane bounding the region must not further divide the simplex. This strictly follows from the fact that the fundamental domain is the minimal volume in a given space upon division by a Coxeter group, or equivalently, upon division by a set of mirrors closed under self-reflection.

Explicitly, for planes P_i and P_j defined by $\langle \vec{x}, \vec{r_i} \rangle = d_i$ and $\langle \vec{x}, \vec{r_j} \rangle = d_j$, the reflection of P_j about P_i should be the hyperplane defined as the locus of any point p in P_j reflected about P_i, and vice-versa. If point q is the image of $p \in P_j$ reflected by P_i, then the midpoint of qp lies in P_i. This allows us to find that P_j reflected by P_i is the plane $P_i\,[P_j]$, given by

$$\langle \vec{x}, R(\vec{r_i})\vec{r_j} \rangle = d_j - 2\frac{\langle \vec{r_i}, \vec{r_j} \rangle}{\langle \vec{r_i}, \vec{r_i} \rangle} d_i, \tag{9}$$

and conversely for P_i reflected about P_j. Equation 9 for the induced plane $P_i\,[P_j]$ can be rewritten as

$$\langle \vec{x}, \vec{r_j} \rangle - d_j = 2\frac{\langle \vec{r_i}, \vec{r_j} \rangle}{\langle \vec{r_i}, \vec{r_i} \rangle} \left(\langle \vec{x}, \vec{r_i} \rangle - d_i \right), \tag{10}$$

and thus we see that if $P_i[P_j]$ is to divide S, i.e. there exist points lying on $P_i[P_j]$ contained in S, then we must have

$$\frac{\langle \vec{r_i}, \vec{r_j} \rangle}{\langle \vec{r_i}, \vec{r_i} \rangle} > 0, \tag{11}$$

or equivalently $\langle \vec{r_i}, \vec{r_j} \rangle > 0$. Thus we can deduce that a bounded fundamental domain that is not further subdivided by reflections about its mirror boundaries (and thus can tile the full space) must have $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$ for each of its distinct roots.

We can thus begin to argue about the existence of various fundamental domains in $\mathbb{E}^n$. First, we shall impose a condition that greatly simplifies the classification of allowed fundamental domains: we shall ask that the mirrors defining our fundamental domain cannot be split into two disjoint sets that are mutually orthogonal. If this were to be true, then the full reflection group generated by the mirrors bounding the domain could be written as the product of the two reflection groups defined by the mutually orthogonal disjoint sets. Thus we shall only study such groups that cannot be decomposed in this way; these groups are called **irreducible**.

Lemma 3 ([1])**.** *In a Euclidean space of dimension n, there are at most $n+1$ vectors $\vec{v}_i$ satisfying $\langle \vec{v}_i, \vec{v}_j \rangle \leq 0$ for $i \neq j$, where the reflection group defined by $\{\vec{v}_i\}$ is irreducible.*

Proof. Suppose there are m vectors $\vec{v}_i$. Wlog, take $m > n$ as we are trying to find an upper bound for m. Also note that it is not hard to construct exactly n such vectors satisfying $\langle \vec{v}_i, \vec{v}_j \rangle < 0$ (for example, $(1, 0, 0, 0, 0, \cdots)$, $(-1, 1, 0, 0, 0, \cdots)$, $(-1, -2, 1, 0, 0, \cdots)$, $(-1, -2, -6, 1, 0, \cdots)$, $(-1, -2, -6, -42, \cdots)$). Now consider the matrix defined by $A_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$ for all $1 \leq i, j \leq m$. Since $A_{ii} \geq 0$, $A_{ij} \leq 0$, and $m > n$, this matrix is positive-semidefinite. Further, the matrix A must be **indecomposible** (also called **connected**), meaning that there is no partition of the set of indices $\{1, \cdots, m\}$ into two nonempty sets I, J such that $A_{ij} = 0$ whenever $i \in I$,$j \in J$. We see immediately that this condition is equivalent to the reflection group defined by $\{\vec{v}_i\}$ being irreducible; $\{\vec{v}_i\}$ would be reducible if and only if it could be split into two sets of roots that are mutually orthogonal. As shown in [1, Section 10.22], every positive-semidefinite indecomposible matrix A has nullity 1, implying that if $m > n$, must have $m = n + 1$. ■

Thus, by considering all possible self-consistent values of each m_{ij} defining an angle between two roots for an arbitrary *finite* point Coxeter group, i.e. $m_{ij} \neq \infty$, Coxeter neatly classified all such irreducible groups in $\mathbb{E}^n$ with n mirrors. These groups precisely correspond to irreducible finite point groups in Euclidean space. They can be described by **Coxeter diagrams**, where each point in the diagram corresponds to a generator in the group, and edges between points are labelled with the corresponding m_{ij} value. By convention, if $m_{ij} = 3$, the "3" is not written on the diagram, and if $m_{ij} = 2$ (perpendicular roots/mirrors), the edge between the corresponding points is omitted. These point group diagrams are given both in Table 1 and 2, which are separated based on the constraints from **Theorem** 2, showing which such point groups are crystallographic. For the crystallographic point groups, we can then construct the corresponding affine group/kaleidoscope group with $n + 1$ mirrors, which gives a Coxeter diagram called an **affine Coxeter diagram**, shown in Table 1. Note that in this classification there are some duplicates, e.g. $I_2(3) = A_2, I_2(5) = H_2, A_3 = D_3, D_2 = A_1 + A_1$.

Remark. There are also affine extensions [37] of the non-crystallographic groups in Table 2, constructed by translating the groups by their **root lattice**, the lattice constructed by integer linear combinations of generating roots. As we've seen in Section 2.1 in the case of H_2 (pentagonal symmetry), this construction of mirrors is not compatible with any lattice. However, by placing a point at the intersections of n mirrors, one can define a point set which that is dense in the space. These sets are related to quasicrystals and other applications in physics, chemistry, and biology [38–41].

Table 1: Coxeter diagrams for irreducible finite (Euclidean) crystallographic groups and their kaleidoscopes. *Using Ben McKay's* `dynkin-diagrams` *Package.*

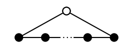
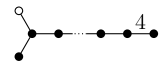
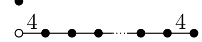
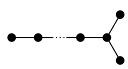
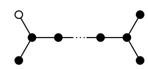
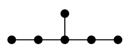
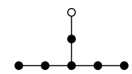
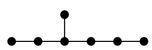
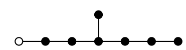
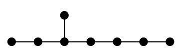
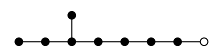
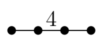
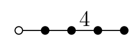
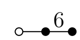
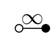
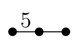
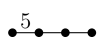
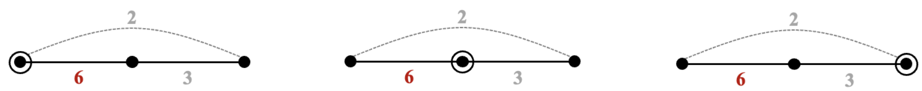
	Point Groups		Affine Groups	
A_n		n generators, $n \geq 1$	$\tilde{A}_n$	
$B_n \equiv C_n$		n generators, $n \geq 2$	$\tilde{B}_n$	
			$\tilde{C}_n$	
D_n		n generators, $n \geq 4$	$\tilde{D}_n$	
E_6		6 generators, exceptional	$\tilde{E}_6$	
E_7		7 generators, exceptional	$\tilde{E}_7$	
E_8		8 generators, exceptional	$\tilde{E}_8$	
F_4		4 generators	$\tilde{F}_4$	
G_2		2 generators	$\tilde{G}_2$	
			$\tilde{I} \equiv \tilde{A}_1$	

Table 2: Coxeter diagrams for irreducible finite non-crystallographic groups

H_2		Point group for pentagon and decagon
H_3		Point group for dodecahedron and icosahedron
H_4		Point group for the 600-cell and 120-cell polychora
$I_2(p)$		Prime $p \neq 3$, irreducible dihedral group

Note that the affine diagrams in Table 1 precisely describe the fundamental region of the corresponding kaleidoscope. It is a polytope bounded by mirrors that intersect at angles π/m_{ij} where m_{ij} is the label of the corresponding edge in the diagram. The unit cell of the lattice stabilised by the affine group is precisely the image of the fundamental polytope under all mirrors going through the origin, or equivalently the mirrors given by the corresponding point group. To connect this back to an earlier example, we recognise that the point group of our hexagonal lattice in Figure 1 is given by diagram G_2, as the angle between the generating planes is $\pi/6$. Further, the fundamental polygon is given exactly by $\tilde{G}_2$, and the previous example of the lattice and honeycomb constructions can be seen clearly from $\tilde{G}_2$ as in Figure 5, where the selected point in the graph corresponds to the point in Euclidean space at the intersection of all but one mirror.



(a) Figure 4a Diagram (b) Figure 4b Diagram (c) Figure 4c Diagram

Figure 5: Wythoff Construction for Lattice and Honeycombs from $\tilde{G}_2$ Kaleidoscope

Finally, we can connect this remarkable formalism back to the question of classifying crystalline symmetry in materials, which can correspond to non-equivalent combinations of Euclidean irreducible crystallographic Coxeter groups (and their subgroups) with $n = 3$. For example, the hexagonal prism is given by the diagram $\bullet\!\overset{6}{-}\!\bullet\ \ \bullet$. The classification of all such symmetries in materials is a complicated and fiddly task (as some materials do not have any reflection symmetry but rather strictly subgroups of reflection groups); that said, the concepts highlighted in this chapter are key for undertaking such a task. The interested reader can find more about this classification in [42].

2.6 Summary

We have now seen a complete set of results providing the mathematical foundations to understand regular discrete order and corresponding symmetry groups in Euclidean space. In the following sections, we shall motivate the interest in extending this study to Lorentzian spaces, and analyse how the results in this section change.

To recap the concepts we have seen, we would like to illustrate a chain of "equivalence" that exists between the various objects considered in this chapter (namely lattices, roots and point groups, kaleidoscope groups, and fundamental domains), and how this chain underlies the Euclidean notion of crystallographic order:

> Beginning with an integral lattice, one can find its roots (and equivalently its reflection point group) by **Theorem** 1. Extending this point group by translating it by any integer linear combination of root vectors produces a kaleidoscope group (which must be crystallographic by construction as the point group came from a lattice). The kaleidoscope group defines a fundamental region of finite non-zero volume. Reflecting a given point of the fundamental region about all mirrors in the kaleidoscope group regenerates the original lattice.

One can begin at any construction in this chain and end up back at the same construction, *so long as* the point group and kaleidoscope group are crystallographic. For example, a non-crystallographic point group can still generate a kaleidoscope group (by translating by integer linear combinations of root vectors) but the kaleidoscope group will be non-crystallographic, so there is no finite and non-zero volume fundamental domain and thus one cannot construct a lattice.

3 Crystallographic vs Kaleidoscopic

Before moving to non-Euclidean spaces, and in particular Lorentzian spaces, we will need to first introduce a related property of reflection groups, which will act as an abstraction and generalisation of the "crystallographic" property (i.e. relating to lattices and Euclidean space) introduced in the previous chapter. The goal of this chapter is to motivate the definition of this property and connect it to the previously-seen Euclidean case.

We begin by more formally analysing the fundamental region P (a subset of Euclidean space) to see how it connects to crystallographic symmetry. Consider T_Λ, the translation group given by vectors in some lattice Λ. Then for $\gamma(\vec{r}, d) \in \Gamma$, and $T_1, T_2 \in T_\Lambda$ translating by $\vec{\ell_1}, \vec{\ell_2} \in \Lambda$, we have

$$\begin{aligned} T_1\gamma &: \vec{x} \mapsto R(\vec{r})\vec{x} + 2d\frac{\vec{r}}{\langle \vec{r}, \vec{r} \rangle} + \vec{\ell_1} \\ \gamma T_2 &: \vec{x} \mapsto R(\vec{r})\vec{x} + 2d\frac{\vec{r}}{\langle \vec{r}, \vec{r} \rangle} + R(\vec{r})\vec{\ell_2}, \end{aligned} \tag{12}$$

where $T_1\gamma \in T_\Lambda\gamma$ and $\gamma T_2 \in \gamma T_\Lambda$. We see that these are the same operation if and only if $R(\vec{r})\vec{\ell_2} \in \Lambda$, where $\vec{\ell_2}$ and $\vec{r}$ were both arbitrary. This means $R(\vec{r})$ must stabilise Λ, and by extension, so must Γ. Recall, a group generated by affine reflections that preserve some lattice is precisely the definition of a crystallographic group. We thus have the following result.

Theorem 3. *A group Γ of affine transformations of n-dimensional Euclidean space is **crystallographic** if and only if it contains a full rank translation group as a normal subgroup. Further, the vectors in that translation group define a rank-n lattice Λ, with a point symmetry group given by the quotient group Γ/T_Λ.*

Proof. The first statement follows since $T_\Lambda \trianglelefteq \Gamma$ if and only if $T_\Lambda \gamma = \gamma T_\Lambda$ for any $\gamma \in \Gamma$ if and only if Γ is crystallographic. Further, cosets $\gamma_1 T_\Lambda$ and $\gamma_2 T_\Lambda$ of Γ/T_Λ are equal when the affine mirrors defined by γ_1, γ_2 are parallel, thus reducing to $\Gamma/T_\Lambda \cong G$, where G is the point symmetry group of Λ. ■

We thus see that the orbit space $V/\Gamma \cong (V/T_\Lambda)/(\Gamma/T_\Lambda)$ for crystallographic Γ. Note V and T_Λ are both abelian so $T_\Lambda \trianglelefteq V$, and the quotient group V/T_Λ identifies vectors in V that aren't separated by a lattice vector, and thus is equivalent to the Wigner-Seitz unit cell. Thus, *the fundamental domain P is also the orbit space of a point group of a lattice on its unit cell.* This can be seen clearly in Figure 4a, which shows that reflecting the fundamental domain about only the mirrors passing through the origin regenerates the Wigner-Seitz hexagonal unit cell from Figure 1b. This once again demonstrates the result of **Lemma** 1 in Section 2.1 that this choice of unit cell has the full point symmetry group of the lattice.

This discussion motivates our definition of kaleidoscopic symmetry: A group G generated by reflections (both through the origin or affine) acting on an inner product space V (not necessarily Euclidean) is **kaleidoscopic in** V if the region V/G has finite and non-zero volume (as is measured in V). We'll also refer to such a V/G region as an orbifold.

Note that this definition precisely recovers the definitions and properties discussed above for Euclidean kaleidoscope groups defined from a lattice i.e. groups that are crystallographic. However, the definition of kaleidoscopic does not require a notion of lattices, as it only depends on specifying a space and a group generated by reflections. That said, we indeed recover a definition consistent with the previously-presented Euclidean results, as Euclidean affine reflection groups are crystallographic if and only if they stabilise a lattice, which happens if and only if the fundamental domain they define is of finite and non-zero volume. As we will see in later chapters, this does not generalise to non-euclidean spaces. Further, we highlight the separation between crystallographic and kaleidoscopic by noting that a finite point group satisfying **Theorem** 2 would be stabilise a lattice (by construction) and thus be crystallographic, but would not be kaleidoscopic as it would divide Euclidean space into unbounded subregions. We'll see in the next chapter that point groups can indeed by kaleidoscopic in other spaces.[5]

[5] As a final note for the interested reader, we remark that the definition of a kaleidoscopic group used in this essay is the same as the definition used in the study of discrete complex groups [43, 44], as well as a somewhat-similar (but separate and more complicated) notion

4 Beyond the Euclidean Case

The goal of this chapter is to investigate non-Euclidean extensions of the formalisms and constructions described in the previous chapter, to see what concepts generalise well. We thus aim to motivate a more general definition of crystallographic symmetry that recovers the Euclidean case definitions and results, but is more applicable to non-Euclidean (and in particular, Lorentzian) discrete symmetry classification.

Remark. Whenever a space is written with only one dimension specified i.e. $\mathbb{R}^n$ and $\mathbb{E}^n$ mean a positive definite Euclidean space. From here on, we will take the convention that timelike coordinates have a negative signature, so $\mathbb{R}^{-q,+p}$ is the Lorentzian space with q negative-definite and p positive-definite components. The inner product on this space, in the standard basis, is given by

$$\langle \vec{v}, \vec{w} \rangle = -t_v^1 t_w^1 - \cdots - t_v^q t_w^q + x_v^1 x_w^1 + \cdots + x_v^p x_w^q.$$

When vector components are written explicitly, the timelike components will come first; e.g. $(2, 1) \in \mathbb{R}^{-1,+1}$ is a timelike vector while $(1, 2)$ is spacelike.

4.1 Spherical Groups and Tilings

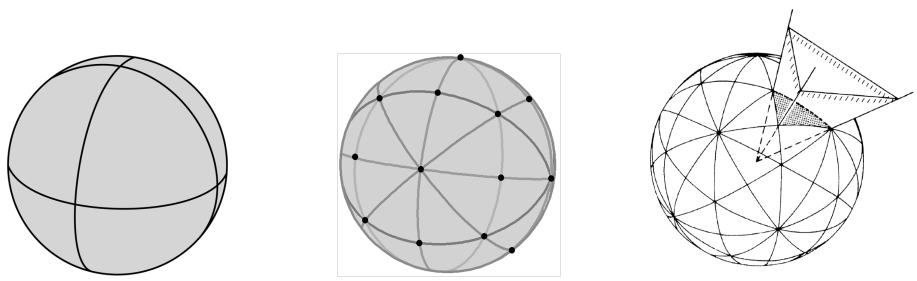
Consider a finite Coxeter point group G and its affine extension $\tilde{G}$ by its root lattice. As G is a finite group, obviously the orbit space $\mathbb{E}^n/G$ is infinite, even though $\tilde{G}$ (being an infinite group) could define a finite volume orbifold $\mathbb{E}^n/\tilde{G}$ so long as $\tilde{G}$ is kaleidoscopic. However, G itself can define a finite volume fundamental domain if instead, it acts on $\mathbb{S}^{n-1}$, the spherical space embedded in $\mathbb{E}^n$ given by $\mathbb{S}^{n-1} = \{\vec{v} \in \mathbb{E}^n \mid \langle \vec{v}, \vec{v} \rangle = 1\}$.

This gives a remarkable and natural extension of discretisations into curved spaces. In Euclidean space, one discretises the space by defining a lattice on that space. However, lattices (of any kind, including Bravais) are no longer a sensible notion in spherical geometry, as there is no "translation" in curved spaces. That said, embedding $\mathbb{S}^{n-1}$ in $\mathbb{E}^n$ provides a well-defined notion of reflections in $\mathbb{S}^{n-1}$, which as we shall see, can define a tiling of $\mathbb{S}^{n-1}$ by a fundamental region, thus giving a non-lattice discretisation that still retains a discrete subgroup of the full isometries of $\mathbb{S}^{n-1}$. Specifically, a reflection in $\mathbb{E}^n$ about a codimension-1 Euclidean hyperplane through the

of "reflective" definition used in [34, 45, 46] to refer to certain nice hyperbolic groups.

origin defines a reflection in $\mathbb{S}^{n-1}$ about the intersection of the Euclidean hyperplanes with $\mathbb{S}^{n-1}$, which is again a codimension-1 surface in $\mathbb{S}^{n-1}$. This does not work for affine reflections as they do not stabilise $\mathbb{S}^{n-1}$.

Just as we've previously seen fundamental domains arrising as subspaces of Euclidean space upon division by an affine reflection group, spherical space can also be subdivided by point reflection groups. Some examples are shown in Figure 6.



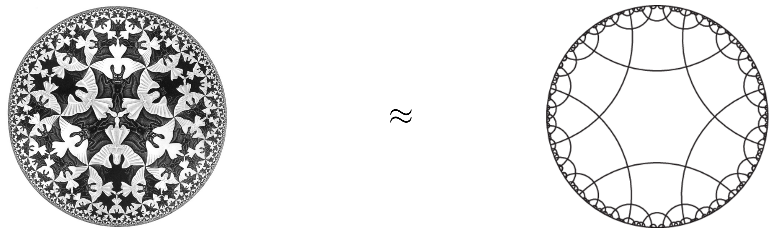
(a) Division by $\bullet \quad \bullet \quad \bullet$ (b) Division by $\bullet - \bullet \overset{4}{-} \bullet$ (c) Division by $\bullet \overset{5}{-} \bullet - \bullet$

Figure 6: Some Spherical Tilings induced by reflection groups. *Figures (a), (b) from Wikipedia; Figure (c) from* [45].

Thus, we that the spherical groups can be thought of as kaleidoscope groups when acting on a sphere. This is an indication that the notion of kaleidoscopic depends on what space one is acting on with a discrete group. Again, we see that while spherical space does not have a sensible notion of a lattices and so crystallographic symmetry does not generalise, kaleidoscopic symmetry still makes sense.

4.2 Hyperbolic Space

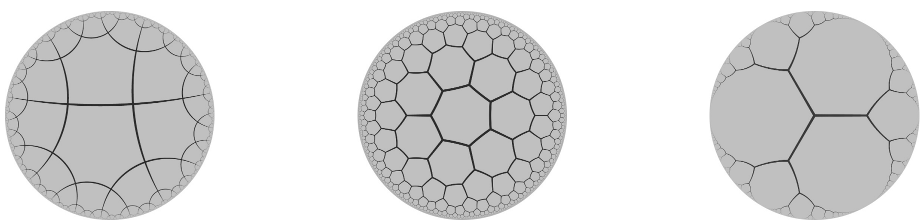
Studies of tilings of hyperbolic space [47, 48] and the corresponding symmetry groups have led to extensive and fascinating work, all of which we cannot comprehensively cover in this essay, and thus this section will be quite terse and not introduce everything in full enough detail. For our purposes, we are interested in introducing hyperbolic space as it has a natural embedding as a surface in Minkowski space, similar to the sphere in Euclidean space. Before we begin, we can see that there are indeed incredibly interesting phenomena to be found when considering tilings of hyperbolic space. A sample of tilings of two-dimensional hyperbolic space can be found in Figure 8, modeled using



Angels and Devils M.C. Escher, 1960. Order-4 Hexagonal Tiling.

Figure 7: Poincare Disc model of $\mathbb{H}^2$. *Figures from Wikipedia*

the Poincare disc as is familiar from the infamous work of M.C. Escher in Figure 7. Note that Figure 8c is particularly strange as it's a regular tiling of hyperbolic space using regions with infinitely many sides, called apeirogons.



(a) Order-4 Pentagonal Tiling (b) Heptagonal Tiling (c) Order-3 Apeirogonal Tiling

Figure 8: Some Hyperbolic Tilings. *Figures from Wikipedia.*

Indeed, although hyperbolic space $\mathbb{H}^n$ is a positive definite space, it can not be represented as a subspace of Euclidean space that is preserved by the isometries of Euclidean space. Instead, $\mathbb{H}^n$ can be embedded in Minkowski space $\mathbb{R}^{-1,+n}$. Considering the set

$$\{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = -1\} = \mathbb{H}^n_+ \cup \mathbb{H}^n_-$$

i.e. two connected components which are each hyperbolic sheets. This can also be written as

$$\begin{aligned} \mathbb{H}^n_+ &= \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = -1, v^t > 0\} \\ \mathbb{H}^n_- &= \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = -1, v^t < 0\}. \end{aligned} \tag{13}$$

When discussing hyperbolic space, we take only one sheet and write $\mathbb{H}^n = \mathbb{H}^n_+$. $\mathbb{H}^n$ is then preserved by all isometries of Minkowski space that preserve the orientation of the future and past lightcone.

To discuss reflections in Lorentzian space, we must return to the definition of reflections in Equation (4). This equation does not change in the Lorentzian case. However, in contrast to Equation (5), the matrix representation of $R(\vec{r})$, the reflection about a mirror normal (with regard to the Lorentzian metric) to the root $\vec{r}$, becomes

$$R(\vec{r})^{\alpha}_{\beta} = \eta^{\alpha}_{\beta} - 2\frac{r_{\beta}r^{\alpha}}{\langle \vec{r}, \vec{r} \rangle}, \tag{14}$$

where $\eta = \mathrm{Diag}(-1, \cdots, -1, +1, \cdots, +1)$ is the metric on $\mathbb{R}^{-q,+p}$.

The symmetries that preserve $\mathbb{H}^n$ are precisely the reflections about hyperplanes through the origin, orthogonal to spacelike normal vectors. This is because hyperplanes normal to spacelike roots must intersect $\mathbb{H}^n$. Conversely, hyperplanes normal to timelike groups do not intersect $\mathbb{H}^n$ and flip the orientation of the space, specifically, interchange the future lightcone:

$$C_+ = \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = 0, v^t > 0\}$$

with the past lightcone

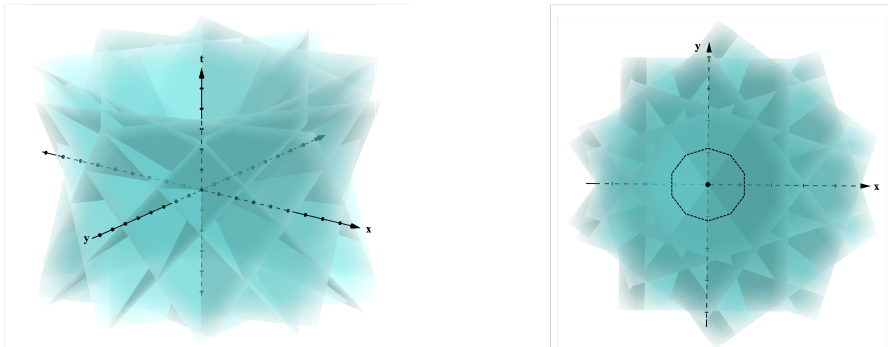
$$C_- = \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = 0, v^t < 0\}$$

and are thus not symmetries of $\mathbb{H}^n$. Further, symmetries of $\mathbb{H}^n$ cannot be affine reflections of $\mathbb{R}^{-1,+n}$ as those transformation would not be isometries of $\mathbb{H}^n$, in the same way that affine reflections in $\mathbb{E}^{n+1}$ are not isometries of $\mathbb{S}^n$.

As we've seen previously in the Euclidean and Spherical cases, point groups of reflections in $\mathbb{R}^{-1,+n}$ can act on $\mathbb{H}^n$ in a way that defines a fundamental domain. Such fundamental domains can be studied, for instance, in two ways. First, by considering the kinds of angles allowed by hyperbolic space. Secondly, by considering mirror hyperplanes in a Lorentzian embedding space with spacelike root vectors $\vec{r}_i$ satisfying $\langle \vec{r}_i, \vec{r}_j \rangle \leq 0$; we've seen this before as the condition for mirrors defined by (spacelike)[6] roots to not further subdivide a region upon reflection. Studying the intersection of such mirrors with $\mathbb{H}^n$ can be done systematically by Vinberg's Algorithm [49], which is used to classify reflection groups that are kaleidoscopic in $\mathbb{H}^n$.

However, we note that **Lemma** 3 giving bounds on the number of mirrors in a Euclidean space satisfying $\langle \vec{r}_i, \vec{r}_j \rangle \leq 0$ does not generalise to indefinite spaces.

[6]This condition was introduced when discussing Euclidean reflections, and all Euclidean roots are spacelike trivially.



(a) Orthographic illustration (b) Looking down from positive t-axis

Figure 9: Illustration of $m = 10$ mirrors in $\mathbb{R}^{-1,+2}$ with spacelike roots satisfying $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$.

Lemma 4. *If a set of spacelike m roots $\vec{v}_i$ have indefinite Span$\{\vec{v}_i\} \cong \mathbb{R}^{-b,+a}$ for $a > 1, b > 0$, then $\vec{v}_i$ can satisfy $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$ and generate an irreducible reflection group for arbitrarily large m.*

Proof. For any given m, there is some ϵ such that $\cos(2\pi/m) < (1-\epsilon)^2$. Then define m spacelike vectors $\vec{v}_k$ using two spatial coordinates and one time coordinate by $x_k^1 = \cos(2k\pi/m), x_k^2 = \sin(2k\pi/m), t_k^1 = 1-\epsilon$ with all remaining coordinates set to zero[7]. Then $\langle \vec{v}_j, \vec{v}_k \rangle = cos(2(j-k)\pi/m) - (1-\epsilon)^2$ and all properties are satisfied. Note that if m is even, then some of the vectors are orthogonal but irreducibility is still satisfied since each vector is still non-orthogonal to $m-2$ other vectors. ■

As seen in **Lemma** 4, once the span of these roots is indefinite, there can be arbitrarily many of them, i.e. corresponding to arbitrarily many mirrors. If all such mirrors pass through the origin, this corresponds to the hyperbolic reflection groups and hyperbolic tilings. This is illustrated in Figure 9, showing an example in $\mathbb{R}^{-1,+2}$ of $m = 10$ mirrors satisfying the required properties and forming an irreducible group, given by the construction in the Proof of **Lemma** 4. The region

$$S = \{\vec{x} \in V \mid \langle \vec{x}, \vec{s_j} \rangle \leq 0 \text{ for all } 1 \leq j \leq m\}$$

defined by this point group will intersect hyperbolic space $\mathbb{H}^2$ in some 10-sided polygonal shape, illustrated in Figure Figure 9b as viewed perpendicular to the positive t-axis.

[7]With thanks to Henry Cohn for discussion of this construction.

4.3 Summary

This chapter has been light on detail and explanation given the extent of available results and fascinating things one could say about non-Euclidean geometries. To summarise, hyperbolic space, having negative curvature, allows for fewer restrictions on the angles of regular polytopes inside of it [12, 13], thus allowing for strange and un-intuitive tilings of the space, some of which are presented in Figure 8. The fact that we can study hyperbolic space as an embedding of Lorentzian space further motivates our study of reflections and reflection groups in Lorentzian space.

5 Initial Investigations in Lorentzian Space

Thus far we have seen an introduction to the extensive investigations of reflective symmetries in hyperbolic space, embedded in Lorentzian space. However, recall that these investigations have been limited to only considering spacelike root vectors (corresponding to timelike mirrors which are symmetries of hyperbolic space). Thus, for this essay's purpose to better understand the geometry of Lorentzian spaces through the lens of regular structures, tilings, and discrete groups, one must also consider timelike roots. Afterall, the full symmetry group of Minkowski space of course is given by Poincare symmetry, not hyperbolic symmetry, and thus we're very interested in probing what mathematical, geometrical, or algebraic insights a full study of reflection groups in Lorentzian space could lead to. Indeed, by the *Cartan-Dieudonné Theorem* [50, 51], the group $O(-q,+p)$ of orthogonal transformations of $\mathbb{R}^{-q,+p}$ can be generated by hyperplane reflections. Indeed, all boosts can be written as an even product of reflections about timelike roots, and more generally, any element in $O(+p,-q)$ can be written as a product of at most $p+q$ hyperplane reflections [52, 53].

In fact, there are quite a few well-known lattices in Lorentzian spaces [45, 54]. This even further motivates questions about their point symmetry groups, affine groups they correspond to and whether they can provide insights towards the question of which tilings (discretisations) have the largest discrete symmetry group in a given Lorentzian space, thus preserving as much geometry of the original space as possible. We have seen in Section 2.3 a criterion for determining the roots corresponding to reflective symmetries of a given lattice. More specifically, the proof of Theorem 1 does not rely on the root being spacelike; the order of the coset $2\vec{v}/\langle\vec{v},\vec{v}\rangle + \Lambda$ in Λ^*/Λ for timelike root $\vec{v}$ is either $|\langle\vec{v},\vec{v}\rangle|$ or $|\langle\vec{v},\vec{v}\rangle|/2$ since $\vec{v}$ is primitive, meaning that even in Lorentzian space we have $\langle\vec{v},\vec{v}\rangle$ dividing $2\det(\Lambda)$. Thus, timelike roots still define valid reflective symmetries of an integral Lorentzian lattice and are worth consideration.

Example. Consider the lattice the lattice consisting of all integer points in $\mathbb{R}^{-1,+1}$, namely $\Lambda = \{\vec{x} \in \mathbb{R}^{-1,+1} \mid \vec{x} = (m,n) \text{ for } n,m \in \mathbb{Z}\} \equiv I_{-1,+1}$ called the **odd unimodular lattice** of signature $(-1,+1)$ where unimodular means its Gram matrix has determinant one[8]. By **Theorem** 1 then, the roots have a norm dividing 2. In an indefinite space, this means roots of norm $\pm1, \pm2$, specifically giving spacelike and timelike roots $(0,1)$ and $(1,0)$ respectively. One can painlessly verify that for $\vec{x} = (m,n) \in I_{(1,1)}$, have $R(1,0)\vec{x} = (-m,n) \in I_{(1,1)}$, so indeed reflection about a timelike root is a well-defined symmetry of the lattice. Also, note that $(0,1)$ and $(1,0)$ are the only roots of the lattice, so it's easy to see that the kaleidoscope group for this lattice consists of all vertical and horizontal mirror lines passing through each integer point in $\mathbb{R}^{-1,+1}$. Thus the fundamental domain is a unit square and its roots are reducible (as $(0,\pm1)$ are orthogonal to $(\pm1,0)$)

Example. The next nicest lattice to consider is the odd unimodular lattice in $\mathbb{R}^{-1,+2}$ given by $I_{-2,+1} = \{\vec{x} \in \mathbb{R}^{-1,+2} \mid \vec{x} = (m,n,\ell) \text{ for } n,m,\ell \in \mathbb{Z}\}$. Again the roots are all lattice vectors with norm $\pm1, \pm2$. However, each of the four Diophantine equations $x_1^2 + x_2^2 - t^2 = \pm1, \pm2$ have infinitely many solutions. Thus even the point group for this lattice is infinite. This is not an uncommon feature for Lorentzian reflection groups, as we'll see below. For the interested reader, there are some nice descriptions of the $-1, -2$ solutions by points of regular and quasiregular hyperbolic tilings [55], but so far we have been unable to find a closed-form solution for all roots of this lattice. It seems the best approach to solve such Diophantine equations for numerical use is to put them in the form of a Pell Equation [56].

Theorem 4. *For a collection of roots corresponding to mirrors through the origin in a Lorentzian space, a point group generated by reflections about these roots is infinite, unless either*

(i) The induced metric in the space spanned by the roots is semi-definite.

[8] $I_{-1,+1}$ is also self-dual as integral unimodular lattices must be self-dual.

(ii) The induced metric in the space spanned by the roots is indefinite, but for each pair of roots that span an indefinite space, such roots are orthogonal.

Proof. Reflections corresponding to these roots will fix all points in the orthogonal complement of the space spanned by the roots, and thus we only have to consider the effect reflections have in this subspace. Consider n roots $r_1, \cdots, r_n \in \mathbb{R}^{(p,q)}$ (where $q > 0$) and define the induced metric on $\mathrm{Span}\{r_1, \cdots, r_n\}$ by

$$g^{\mathbf{R}}_{ab} = \eta^{(-q,+p)}_{\mu\nu} r^{\mu}_a r^{\nu}_b \tag{15}$$

Consider the signature of the induced metric (i.e. given by the eigenvalues of). If the induced metric is definite (or semi-definite), then the group generated is equivalent to a Euclidean reflection group. If the metric is instead indefinite, then there exists a frame such that two of the roots, say $\vec{r}_1$ and $\vec{r}_2$, span an indefinite rank-2 space. If they are also not orthogonal, then we can write, for some hyperbolic angle $\chi \neq 0$ and constants $a, b \in \mathbb{R}$,

$$\begin{array}{lll} \vec{r}_1 = \sqrt{a}(0,1,0,\cdots,0) & \vec{r}_2 = \sqrt{b}(\sinh\chi, \cosh\chi, 0, \cdots, 0) & \vec{r}_1,\ \vec{r}_2 \text{ spacelike} \\ \vec{r}_1 = \sqrt{a}(1,0,0,\cdots,0) & \vec{r}_2 = \sqrt{b}(\sinh\chi, \cosh\chi, 0, \cdots, 0) & \vec{r}_1 \text{ timelike, } \vec{r}_2 \text{ spacelike} \\ \vec{r}_1 = \sqrt{a}(1,0,0,\cdots,0) & \vec{r}_2 = \sqrt{b}(\cosh\chi, \sinh\chi, 0, \cdots, 0) & \vec{r}_1,\ \vec{r}_2 \text{ timelike} \end{array}$$

where these vectors are written in a component form s.t. the first component is a time-coordinate and the second is a space-coordinate. For each of the above three cases, reflecting these roots about each other generates the following additional roots respectively:

$$\begin{array}{lll} \sqrt{a}(0,-1) & \sqrt{b}\,(\pm\sinh(2m-1)\chi, \pm\cosh(2m-1)\chi) & \sqrt{a}\,(\pm\sinh 2m\chi, \pm\cosh 2m\chi) \\ \sqrt{a}(-1,0) & \sqrt{b}\,(\pm\sinh(2m-1)\chi, \pm\cosh(2m-1)\chi) & \sqrt{a}\,(\pm\sinh 2m\chi, \pm\cosh 2m\chi) \\ \sqrt{a}(-1,0) & \sqrt{b}\,(\pm\cosh(2m-1)\chi, \pm\sinh(2m-1)\chi) & \sqrt{a}\,(\pm\cosh 2m\chi, \pm\sinh 2m\chi) \end{array}$$

for any integer $m \geq 1$, where components outside of the rank-2 span have been omitted. Thus the group is infinite. ■

Given that the consideration of timelike roots leads almost certainly to infinite reflection groups, it is then very natural to ask how one can consider kaleidoscopic symmetry in such cases. For example, in the case of the roots stabilising $I_{-2,+1}$, it seems incredibly unclear how the affine extension of such a dense reflection group could divide the space into regions of non-zero volume. And yet, this group must be crystallographic by construction as it stabilises the lattice $I_{-2,+1}$. This is our final and most striking piece of evidence for the differences between crystallographic and kaleidoscopic symmetry: not only is kaleidoscopic symmetry a generalisation that allows for curved spaces, but it is also necessary in Lorentzian spaces (even though they are flat) due to the Euclidean equivalence of crystallographic and kaleidoscopic breaking down.

Recall, as in Chapter 2, the fundamental domain is required to have minimal volume and therefore can not be further subdivided by reflections. The induced mirror $P_i[P_j]$ will divide a region bounded by mirrors P_i and P_j when

$$\frac{\langle \vec{r_i}, \vec{r_j} \rangle}{\langle \vec{r_i}, \vec{r_i} \rangle} > 0$$

or equivalently $\langle \vec{r_i}, \vec{r_j} \rangle > 0$ if r_i is spacelike and $\langle \vec{r_i}, \vec{r_j} \rangle < 0$ if r_i is timelike. Immediately we can deduce constraints on the roots of mirrors bounding a fundamental domain in a Lorentzian space:

A. If $\vec{r_i}$ is timelike and $\vec{r_j}$ is spacelike, then for $P_j\,[P_i]$ to not further divide the domain, we must have $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$. But conversely, must have $\langle \vec{r_i}, \vec{r_j} \rangle \geq 0$ for $P_i\,[P_j]$ to not further divide the domain. Thus requires $\langle \vec{r_i}, \vec{r_j} \rangle = 0$.

B. If both $\vec{r_i} \neq \vec{r_j}$ are timelike, then must have $\langle \vec{r_i}, \vec{r_j} \rangle \geq 0$ so that neither $P_j\,[P_i]$ nor $P_i\,[P_j]$ further divide the domain.

C. If both $\vec{r_i} \neq \vec{r_j}$ are spacelike, then must have $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$.

We have seen in **Lemma** 4 that Property C can be made to hold for arbitrarily many spacelike roots spanning an indefinite subspace. The same argument holds instead for timelike vectors which span an indefinite subspace with at least two negative eigenvalues in the induced metric, by constructing roots $x^1_k = 1 - \epsilon, t^1_k = \cos(2k\pi/m), t^2_k = \sin(2k\pi/m)$ as in **Lemma** 4. In Appendix A, we utilise this to begin arguing about various fundamental domains in Lorentzian spaces from Properties A-C above.

5.1 Future Directions

The formalism introduced so far provides essential first steps in our goal of reasoning about kaleidoscopic Lorentzian symmetry, but it is not yet sufficient. Properties A-C only impose restrictions on the roots, and not on more complicated behaviour that could arise from a full consideration of *affine* mirrors. Additionally, these properties only guarantee that a certain region won't be further divided upon all allowed reflections, but this does not apply to the space outside of that region. This would require carefully utilising **Theorem** 4, which we have not yet done in this discussion.

We posit that, *unless any pair of mirrors whose roots span an indefinite subspace are orthogonal, an affine reflection group in Lorentzian space cannot be kaleidoscopic.* We have not yet proved this, but we argue it heuristically as follows: If an affine reflection group were to be kaleidoscopic in a Lorentzian space, it would have to be defined by roots spanning an indefinite subspace, since if the roots were to only span a definite subspace, the orbifold of such a group acting on an indefinite space would not have finite volume. As soon as any two roots span an indefinite subspace *and* are not orthogonal, they generate infinitely many reflections by **Theorem** 4. As we're considering an affine group, we then argue that we must have two such copies of infinitely-many generated mirrors each passing through two such points that are only finitely separated. Thus the space between those two such points would be subdivided by the group action into zero volume, and hence the affine group cannot be kaleidoscopic.

If the statement in the paragraph above were to be true, it would then reduce Lorentzian reflection groups to either 1) only point groups passing through the origin or 2) orthogonal products of two Euclidean kaleidoscopes (one of positive signature, the other of negative signature) and a choice of full rank positive/negative orthogonal Euclidean subspaces to place the aforementied Euclidean kaleidoscopes.

We emphasise that the above argument is not rigourous, and a careful study of allowed choices of mirrors, distances between mirrors, and the interaction of these choices with **Theorem** 4 is certainly needed. We hope to accomplish this in future work.

6 Conclusion

This essay began as an endeavour to understand what, if any, strange and fundamental mathematics and geometry would result from a study of kaleidoscope groups and kaleidoscopic symmetry in Lorentzian spaces. Such a study is greatly inspired by the compact yet immensely applicable theories pioneered by Coxeter, Cartan, Dynkin, Weyl, Vinberg, and others in the positive definite spaces of constant curvature. We have reviewed and motivated such classical results in Sections 2 and 4, while motivating and clarifying definitions and consequences of kaleidoscopic symmetry in Section 3 towards the goal of generalising to the Lorentzian case. We argue that kaleidoscopic symmetry defined by finite non-zero volume orbifolds of reflection groups is the best such generalisation, and illustrate that while crystallographic and kaleidoscopic symmetry is equivalent in Euclidean spaces, this is no longer true in Lorentzian space. We then present the beginnings of an argument towards classifying kaleidoscopic reflection groups in Lorentzian spaces in Section 5, arguing from minimal principles inspired by the Euclidean treatment.

To conclude, there are two contradictory phenomena at work in Lorentzian spaces, making the study of kaleidoscopic symmetry both difficult and exciting - on the one hand, as we have shown, most Lorentzian point reflection groups have infinite order, and further, timelike and spacelike mirrors must be orthogonal in order to avoid further subdividing a given region bounded by those mirrors. Both of these imply sharp restrictions on allowed kaleidoscopic groups. On the other hand, strange and geometrically non-intuitive things can occur in Lorentzian space due to the indefinite inner product, such as the fact that arbitrarily many mirrors (which still generate an irreducible group) can define a region in Lorentzian space that is not further subdivided. This indicates that there may still be potential for interesting and nontrivial kaleidoscopic discretisations to exist in Lorentzian spaces. Contention of these two seemingly contradictory phenomena warrants further study to answer the question - what is the most symmetry we can give to discrete spacetimes?

7 Acknowledgements

Figure 10: Otter Inside a Kaleidoscope! *Generated by* `hotpot.ai`

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A Lorentzian Kaleidoscopic Restriction

We'll begin our discussion with $\mathbb{R}^{-1,+n}$ and then see how much (or how little) generalises to $\mathbb{R}^{-q,+p}$ for $q > 1$. Following Coxeter's analysis, we will also try to argue about irreducible fundamental domains. Note that for a mix of spacelike and timelike vectors, Property A forces the full reflection group generated by reflections about the mirrors of the domain to be reducible. Thus, in this case, we will work with the next strongest constraint, namely that the spacelike roots and timelike roots separately define irreducible groups, so that the full group is minimally reducible. There are four subcases.

1. Only Timelike Roots

The roots of the mirrors bounding the fundamental domain are only timelike. Property A is satisfied trivially and Property C is irrelevant; let us consider how many timelike vectors can satisfy Property B in a space with only one time dimension. Any timelike vector lies in either the future cone or the past cone. (Note that this doesn't generalise to a space with more than one timelike component as seen in Lemma 4, since the set $\{\vec{x} \mid \langle \vec{x}, \vec{x} \rangle < 0\}$ becomes a single connected component rather than two). For two distinct timelike vectors to have a positive value of the inner product with each other, they must lie in separate cones. Thus we see that there cannot be more than two timelike vectors in $\mathbb{R}^{-1,+n}$ satisfying Property A. Thus the mirrors defining the fundamental domain in a Minkowski space cannot all be timelike, as two mirrors are not enough to bound a finite volume in $n+1$ dimensional space (unless adversarially $n = 0$, but this would then not be a Lorentzian space).

This subcase becomes markedly more complicated when there is more than one time dimension. The argument above does not generalise, and as we've seen from Lemma 4, one can have arbitrarily many timelike vectors forming an irreducible group and satisfying Property B. We hope to investigate this case more in future work, particularly as it relates to AdS^n space embedded in $\mathbb{R}^{-2,+(n-1)}$, and potential induced discretisations of AdS^n from reflection groups in the embedding space with well-defined fundamental domain (i.e. won't be reduced to 0 volume upon reflection by mirror boundaries).

2. Only Spacelike Roots

The roots of the mirrors bounding the fundamental domain are only spacelike. If they span a positive definite space, then by Lemma 3 there can be at most $n+1$ mirrors. This is not enough mirrors to bound a finite volume in an $n+1$ dimensional space (a simplex would require $n+2$). Thus if there are only spacelike roots defining the fundamental domain, they must span an indefinite space.

3. Both Timelike and Spacelike Roots w/ Definite Span

The roots are both timelike $\vec{t_1}, \cdots, \vec{t_a}$ and spacelike $\vec{s_1}, \cdots, \vec{s_b}$, with their spans $\mathrm{Span}\{\vec{t_i}\}$ and $\mathrm{Span}\{\vec{s_j}\}$ being strictly negative and positive definite spaces respectively. Then Property A requires these subsets to be orthogonal. Further, must have $a = 1+1 = 2$ and $b = n+1$ by analogy with the Euclidean cases, as $\mathrm{Span}\{\vec{t_i}\}$ and $\mathrm{Span}\{\vec{s_j}\}$ can be treated are Euclidean. A more in-detail discussion for this subcase can be found in below. This case thus reduces to the orthogonal product of $\tilde{A}_1$ with an $n+1$ mirror Euclidean affine group, and a choice of n-dimensional spacelike subspace.

Further, this subcase indeed generalises to a more general Lorentzian space $\mathbb{R}^{-q,+p}$. In this case, once again a description of the full reflection group reduces to a choice of an Euclidean affine group of q dimensions and p dimensions, and a choice of p-dimensional Euclidean subspace. The respective affine groups are orthogonal.

Proof. We consider a general Lorentzian space $\mathbb{R}^{(+p,-q)}$.

Suppose the mirrors are such that there are $a > 0$ timelike roots $\vec{t_1}, \cdots, \vec{t_a}$ and $b > 0$ spacelike roots $\vec{s_1}, \cdots, \vec{s_b}$ where the spaces $\mathrm{Span}\{\vec{t_i}\}$ and $\mathrm{Span}\{\vec{s_j}\}$ are definite or semidefinite. Then Property A easily gives that these spaces are orthogonal. Further, we now have that the dimension of $\mathrm{Span}\{\vec{t_i}\}$ is bounded by the number of negative definite coordinates i.e. $r_a \leq q$; likewise $r_b \leq p$. By Property C, $\langle \vec{s_i}, \vec{s_j} \rangle \leq 0$ for all $1 \leq i \neq j \leq a$. By Lemma 3, the maximum number of vectors $\vec{t_i}$ that satisfy this property in a positive definite flat space of rank r_a is $r_a + 1$, so $b \leq r_b + 1$. Utilising Property B, we also see that $a \leq r_a + 1$ follows from Lemma 3, as the condition $\langle \vec{t_i}, \vec{t_j} \rangle \geq 0$ for timelike vectors is the negative-definite analog of the conditions in Lemma 3, and so the result follows by considering the matrix $-A$ instead of A.

We can now consider various cases for r_a, r_b. If $r_a < q$ and $r_b < p$ then

$a \leq q$ and $b \leq p$ by above, so the total number of mirrors is $a + b \leq n$ which cannot define a bounded domain in n-dimensional space. Further, taking $r_a = q - 1 < q$ and $r_b = p$ gives at most $n + 1$ mirrors. This is enough mirrors to bound a domain so long as any n of them are linearly independent, but that is not the case here, since the set of $a + r_b = n$ mirrors has rank $r_a + r_b < n$. The same argument holds for $r_a = q$ and $r_b = p - 1 < p$. ■

4. Both Timelike and Spacelike Roots w/ Indefinite Span

The roots are both timelike $\vec{t}_1, \cdots, \vec{t}_a$ and spacelike $\vec{s}_1, \cdots, \vec{s}_b$, but their spans $\mathrm{Span}\{\vec{t}_i\}$ and $\mathrm{Span}\{\vec{s}_j\}$ are allowed to be indefinite. If $\mathrm{Span}\{\vec{s}_j\}$ were to be indefinite then there would be no remaining time components to define $\mathrm{Span}\{\vec{t}_i\}$ and so this subcase reduces to Case 2. If $\mathrm{Span}\{\vec{t}_i\}$ is indefinite, then by the same argument as in Case 1, there can only be at most two timelike roots. For there to be at least $n+2$ mirrors total, would need $\mathrm{Span}\{\vec{s}_j\}$ to be a rank $n - 1$ space. Thus can have <u>two timelike roots spanning $\mathbb{R}^{-1,+2}$ and n irreducible spacelike roots spanning $\mathbb{R}^{n-1}$ orthogonal to the timelike roots.</u> Finally, we note that this subcase becomes again markedly more complicated with more than one timelike dimension.